Limit Superior & Limit Inferior

Def: Given a bounded sequence $\{x_n\}$, define

$$\limsup_{n\to\infty} x_n := \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n .$$

· Define $y_k := \sup_{n\geq k} x_n$. Then y_k is monotonely decreasing.

& y_k is bounded from below (since $\{x_n\}$ is bounded).

By Monotone Convergence Theorem,

the limit of $\{y_k\}$ exists & $\lim_{k\to\infty} y_k = \inf_{k\in\mathbb{N}} y_k$.

i.e. $$\limsup_{n\to\infty} x_n := \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n = \lim_{k\to\infty}\left(\sup_{n\geq k} x_n\right)$$

Similarly,

$$\liminf_{n\to\infty} x_n := \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n = \lim_{k\to\infty}\left(\inf_{n\geq k} x_n\right)$$

Example: Define

$$x_n := \begin{cases} 1+\frac{1}{n} & \text{if } n \text{ is odd,} \\ -\frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

Determine the values of

$$\limsup(x_n) \quad \& \quad \liminf(x_n)$$
$$\sup(x_n) \quad \& \quad \inf(x_n).$$

$$x_n = \left(2, -\frac{1}{2}, \frac{4}{3}, -\frac{1}{4}, \frac{6}{5}, -\frac{1}{6}, \ldots\right).$$

Observe that $\{x_{2k+1}\}$ is deceasing & $x_{2k+1} > 1 \ \forall k$

while $\{x_{2k}\}$ is increasing & $x_{2k} < 0 \ \forall k$.

Let $y_k := \inf\{x_n : n \geq k\}$.

<u>Case 1</u>: k is even <u>Claim</u>: $y_k = x_k$.

· When $n \geq k$ is even, we have

$$x_{k+1} \leq x_n \text{ sinle } \{x_{2k}\} \text{ is increasing.}$$

· When $n > k$ is odd, we have

$$x_{k+1} < 0 < 1 < x_n.$$

Therefore, $y_k = x_{k+1}$ in this case.

<u>Case 2</u>: k is odd. <u>Cluim</u>: $y_k = x_{k+1}$.

· First, we have $x_{k+1} < 0 < 1 < x_k$.

Note that $y_k = \min\{x_k, y_{k+1}\}$ and by case 1,

we have $y_{k+1} = x_{k+1}$ since $k+1$ is even.

Hence, $y_k = x_{k+1}$ when k is odd.

In conclusion,

$$y_k = \begin{cases} x_k, & k \text{ is even} \\ x_{k+1}, & k \text{ is odd} \end{cases} = \begin{cases} -\frac{1}{k}, & k \text{ is even} \\ -\frac{1}{k+1}, & k \text{ is odd.} \end{cases}$$

- Therefore, $\liminf_{n\to\infty}(x_n) = \sup_{k\to\infty} y_k = 0$. (Check)

- $\inf_{n\in\mathbb{N}}\{x_n\} = x_2 = -\frac{1}{2}$.

Define $w_k := \sup\{x_n : n \geq k\}$.

With similar arguments, we have

$$w_k = \begin{cases} x_k, & k \text{ is odd} \\ x_{k+1}, & k \text{ is even} \end{cases} = \begin{cases} 1+\frac{1}{k}, & k \text{ is odd.} \\ 1+\frac{1}{k+1}, & k \text{ is even.} \end{cases}$$

- Therefore, $\limsup_{n\to\infty} x_n = \inf_{k} w_k = 1$ (Check)

- $\sup_{n\in\mathbb{N}}\{x_n\} = x_1 = 2$.

Exercise:

Let $\{x_n\} \subset \mathbb{R}$ be positive. Suppose $\limsup_n \frac{x_{n+1}}{x_n} = a$.

Show that $\limsup_n \sqrt[n]{x_n} \le a$.

Indeed, we have

$$\liminf_n \frac{x_{n+1}}{x_n} \le \liminf_n \sqrt[n]{x_n} \le \limsup_n \sqrt[n]{x_n} \le \limsup_n \frac{x_{n+1}}{x_n}.$$

Proof: We only prove the last inequality.

Since $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = \inf_{k\in\mathbb{N}} \sup_{n\ge k} \frac{x_{n+1}}{x_n} = a$.

Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\sup_{n\ge N} \frac{x_{n+1}}{x_n} < a+\varepsilon$,

i.e. $\frac{x_{n+1}}{x_n} < a+\varepsilon \ \forall n \ge N$.

Hence $\frac{x_n}{x_N} = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_{N+1}}{x_N} < (a+\varepsilon)^{n-N} \quad \forall n > N$.

Thus $\sqrt[n]{x_n} < (a+\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{x_N} = (a+\varepsilon) \cdot \Big(\underbrace{(a+\varepsilon)^{-N} x_N}_{\text{constant}}\Big)^{\frac{1}{n}}$.

Note that

$$\limsup_n \sqrt[n]{x_n} \le (a+\varepsilon) \cdot \limsup_n \left((a+\varepsilon)^{-N} x_N\right)^{\frac{1}{n}} = a+\varepsilon.$$

Here we use the fact that

$$\lim_{n} c^{\frac{1}{n}} = 1 \text{ for some constant } c>0.$$

Since ε is arbitrary, we have

$$\limsup_{n} \sqrt[n]{x_n} \leq a = \limsup_{n} \frac{x_{n+1}}{x_n}.$$